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ON THE EQUATIONS OF MAGNETOELASTIC THIN PLATES

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Hypotheses relative to the character of variation of the electromagnetic field and elastic displacements over the thickness of a plate were formulated in [1, 2] on the basis of solutions obtained by the method of asymptotic integration of the three-dimensional equations of magnetoelasticity. Two-dimensional equations of magnetoelasticity, in which unknown boundary values of the components induced by the electromagnetic field enter, have been obtained on the basis of these hypotheses. The equations obtained must hence be examined in combination with the Maxwell equations for the medium surrounding the plate under general boundary conditions at the interface of the two media. This means that the magnetoelasticity problem nevertheless remains three-dimensional.

On the basis of the mentioned hypotheses for the magnetoelasticity of thin bodies [1, 2], an attempt is made in this paper to reduce the three-dimensional magnetoelasticity problem to a two-dimensional problem, which will substantially facilitate the investigation of questions about the magnetoelasticity of thin bodies.

1. Let an isotropic plate of constant thickness $2h$, fabricated from a material with finite electrical conductivity, be in an external stationary magnetic field with a given magnetic induction vector $\mathbf{B}_0 = (B_{0x}, B_{0y}, B_{0z})$. The problem is solved under the assumption that the Maxwell equations for a vacuum are valid for the medium surrounding the plate. It is also assumed that the influence of displacement currents on the elastic vibrations characteristics can be neglected.

The elastic and electromagnetic properties of the plate material are characterized by the elastic modulus E, the Poisson's ratio ν, the density ρ, the electrical conductivity σ, the magnetic permittivity μ and the dielectric constant ε.

An x, y, z rectangular coordinate system is selected so that the xy coordinate plane coincides with the middle plane of the plate.

The magnetoelasticity hypothesis of thin bodies proposed in [1, 2] is written as

$$u_x = u - z\frac{\partial w}{\partial x}, \quad u_y = v - z\frac{\partial w}{\partial y}, \quad u_z = w(x, y, t) \tag{1.1}$$
$$e_x = \varphi(x, y, t), \; e_y = \psi(x, y, t), \; h_z = f(x, y, t)$$

Here $u = u(x, y, t)$, $v = v(x, y, t)$, $w = w(x, y, t)$ are the sought tangential and normal displacements of points of the plate middle surface, (u_x, u_y, u_z) are the displacements of an arbitrary point of the plate, φ, ψ are the sought tangential components of the induced electric field, f is the sought normal component of the magnetic field induced in the plate. The remaining components h_x, h_y of the induced magnetic field and e_z of the induced electric field are expressed by means of the six sought functions $u, v, w, \varphi, \psi, f$ [2].

The system of equations to determine the sought functions $u, v, w, \varphi, \psi, f$ which has been obtained on the basis of the relationships (1.1), is presented in [1, 2].

The support condition for the plate edges and the continuity conditions for the electromagnetic field components e_x, e_y and h_z on the plate surface and endfaces must be appended to this system of differential equations. In particular, the latter conditions for the rectilinear edge $x=$const are [2]

$$\varepsilon\varphi = e_x^{(e)}|_{z=0}, \quad \psi = \left[e_y^{(e)} + \frac{\mu-1}{\mu}B_{0z}^{(e)}\frac{\partial u}{\partial t}\right]_{z=0} \tag{1.2}$$
$$f = \left[h_z^{(e)} - \frac{\mu-1}{\mu}B_{0x}^{(e)}\frac{\partial w}{\partial x}\right]_{z=0}$$

where $h_z^{(e)}$, $e_x^{(e)}$, $e_y^{(e)}$ are the corresponding components of the electromagnetic field in the external domain.

2. Values of the electromagnetic field components induced on the surface bounding the plate must be available for a complete determination of the displacements and the elecgromagnetic field in the plate. Hence, the equations presented in [1, 2] to determine $u, v, w, \varphi, \psi, f$ must be examined in combination with the Maxwell equations for the external medium

$$\operatorname{rot} \mathbf{h}^{(e)} = 0, \quad \operatorname{div} \mathbf{h}^{(e)} = 0 \tag{2.1}$$

under the following conditions on the plate surface $z = \pm h$ [2]:

$$h_z^{(e)} = \mu f(x, y, t) - (\mu-1)B_{0x}^{(e)}\frac{\partial w}{\partial x} - (\mu-1)B_{0y}^{(e)}\frac{\partial w}{\partial y} \tag{2.2}$$
$$h_x^{(e)} = h_x - \frac{\mu-1}{\mu}B_{0z}^{(e)}\frac{\partial w}{\partial x}, \quad h_y^{(e)} = h_y - \frac{\mu-1}{\mu}B_{0z}^{(e)}\frac{\partial w}{\partial y}$$

The problem of determining the magnetic field components in the medium surrounding the plate then reduces to solving (2.1) with the conditions (2.2) and conditions of the form (1.2) on the plate side surface, as well as conditions for damping of the disturbances at infinity.

The problem of determining the magnetic field $\mathbf{h}^{(e)}$ is solved comparatively simply if the quantity $h_z^{(e)}$ is given in the $z = \pm h$ planes outside the plate. For example, this holds when the plate makes contact over its whole contour with an ideally conducting diaphragm whose motion is given. Then [3]

$$\mu h_z = \mathrm{rot}_z\,(\mathbf{u}_0 \times \mathbf{B}_0),\ (x,\ y) \in \Omega,\ |z| \leqslant h$$

where $\mathbf{u}_0\,(u_{0x}, u_{0y}, u_{0z})$ is a given displacement vector for points of the diaphragm and Ω is the domain in the $z = 0$ plane bounded by the plate contour. Hence, in particular, we obtain $h_z = 0$ in the domains $(x, y) \overline{\in} \Omega$, $|z| \leqslant h$ for a fixed diaphragm ($\mathbf{u}_0 = 0$).

In this case, inserting the potential function Φ by means of

$$\mathbf{h}^{(e)} = \mathrm{grad}\,\Phi \tag{2.3}$$

reduces the problem to the following external Neumann problem for the function Φ:

$$\Delta\Phi = 0 \tag{2.4}$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z\to\pm h} = F_\pm(x, y, t) = \begin{cases} g_\pm(x, y, t), & (x, y) \in \Omega \\ q_\pm(x, y, t), & (x, y) \overline{\in} \Omega \end{cases}$$

$$g_\pm(x, y, t) = \mu f(x, y, t) - (\mu - 1)\left[B_{0x}^{(e)}\frac{\partial w}{\partial x} + B_{0y}^{(e)}\frac{\partial w}{\partial y}\right]_{z=\pm h}$$

$$q_\pm(x, y, t) = \frac{1}{\mu}\left[\mathrm{rot}_z\,(\mathbf{u}_0 \times \mathbf{B}_0)\right]_{z=\pm h}$$

It is known from potential theory that the solution of (2.4) can be represented as the potential of a simple layer (the upper sign is here taken for $z > h$, and the lower for $z < -h$)

$$\Phi(x, y, z, t) = \mp\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\frac{F_\pm(\xi, \eta, t)\,d\xi\,d\eta}{[(x-\xi)^2 + (y-\eta)^2 + (z\mp h)^2]^{1/2}} \tag{2.5}$$

In the particular case when the plate vibrations mode is a cylindrical surface $z = w\,(x, t)$ (plane problem), the solution of the Neumann problem is represented by means of the logarithmic potential of a simple layer.

By virtue of (2.3) and (2.4), we find from (2.5)

$$\frac{h_x^+ - h_x^-}{2} = \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\frac{x-\xi}{\zeta^3}F(\xi, \eta, t)\,d\xi\,d\eta \tag{2.6}$$

$$\frac{h_y^+ - h_y^-}{2} = \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\frac{y-\eta}{\zeta^3}F(\xi, \eta, t)\,d\xi\,d\eta$$

$$F = \begin{cases} \mu f(x, y, t) - (\mu - 1)\left[c_x^+\dfrac{\partial w}{\partial x} + c_y^-\dfrac{\partial w}{\partial y}\right], & (x, y) \in \Omega \\ \dfrac{1}{2\mu}\{[\mathrm{rot}_z\,(\mathbf{u}_0 \times \mathbf{B}_0)]_{z=h} + [\mathrm{rot}_z\,(\mathbf{u}_0 \times \mathbf{B}_0)]_{z=-h}\}, & (x, y) \overline{\in} \Omega \end{cases}$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2,\ c_x^+ = \frac{B_{0x}^{(e)+} + B_{0x}^{(e)-}}{2},\quad c_y^- = \frac{B_{0y}^{(e)+} + B_{0y}^{(e)-}}{2}$$

The integral in (2.6) is understood to be a Cauchy principal value.

Substituting (2.6) into the system presented in [1, 2], we obtain the governing system of equations in the desired functions $u, v, w, \varphi, \psi, f$ of the problem. Therefore, the problem of magnetoelastic plate vibrations reduces in this case to a system of singular integro-differential equations with a Cauchy kernel.

Let us present the mentioned system in the case of the plane problem, when the plate is a strip of width $2a$ in a permanent external magnetic field whose intensity vector is parallel to the x-axis (for simplicity it is assumed that $\mu = 1$, $\mathbf{u}_0 = 0$)

$$\frac{\partial \psi}{\partial x} + \frac{1}{c}\frac{\partial f}{\partial t} = 0 \tag{2.7}$$

$$\frac{\partial f}{\partial x} + \frac{4\pi\sigma}{c}\left(\psi + \frac{B_{0x}}{c}\frac{\partial w}{\partial t}\right) = \frac{1}{\pi h}\int_{-a}^{a}\frac{f(\xi, t)}{x - \xi}\,d\xi$$

$$D\frac{\partial^4 w}{\partial x^4} + 2\rho h\frac{\partial^2 w}{\partial t^2} + \frac{2\sigma h}{c}B_{0x}\left(\psi + \frac{B_{0x}}{c}\frac{\partial w}{\partial t}\right) = 0$$

The integral in (2.7) is also understood in the principal value sense. In the case of an infinite plate the domain of integration will be $(-\infty, \infty)$.

Let us turn to the general case when the values of $h_z^{(e)}$ are unknown along the whole xy plane. In this case a considerably more complex boundary value problem must be solved.

Let us consider the case of the plane problem by replacing the plate by the slit $[-a, a]$ on the real axis. Let the plate move in a complex $z = x + iy$ plane. The normal magnetic field component $h_y^{(e)}$ should satisfy the Laplace equation for two independent variables upon compliance with the conditions

$$h_y^{(e)} = \begin{cases} g_+(x, t) & \text{for } y = +0, \; -a \leqslant x \leqslant a \\ g_-(x, t) & \text{for } y = -0, \; -a \leqslant x \leqslant a \end{cases} \tag{2.8}$$

and the conditions for damping the disturbances at infinity.

The problem therefore is reduced to the following: find a function $W(z)$ which is analytic outside the segment $(-a, a)$, equal to zero at infinity, and whose imaginary part $h_y^{(e)}$ on the upper and lower edges of this segment take on the given values (2.8).

The solution of this problem is [4]

$$h_y^{(e)} = \operatorname{Im}\left\{\frac{1}{2\pi}\int_{-a}^{a}\frac{g_+ - g_-}{\xi - z}\,d\xi - \frac{1}{2\pi i}\sqrt{\frac{z + a}{z - a}}\int_{-a}^{a}\frac{g_+ + g_-}{\xi - z}\sqrt{\frac{a - \xi}{a + \xi}}\,d\xi\right\}$$

We hence find $\partial h_y^{(e)} / \partial y$ and taking into account that the condition $\operatorname{div}\mathbf{h} = 0$ is valid on the plate surfaces ($y = \pm 0$), we determine the value of the quantity $\partial(h_x^+ - h_x^-) / \partial x$. Then by substituting the value found for $\partial(h_x^+ - h_x^-) / \partial x$ into the system from [1, 2], we obtain the governing system of the problem for the desired functions u, w, ψ, f.

Boundary conditions on the plate endfaces for the electromagnetic field components and the usual support conditions for the plate edges must be appended to the governing equations in order to solve specific boundary value problems.

3. As an illustration, let us apply the method elucidated to solve the problem of vibrations of an infinite plate with constant finite electrical conductivity in the presence of an external magnetic field with intensity vector parallel to the x-axis.

We seek the solution of the system (2.7) as waves being propagated along the x-axis. Then substitution into (2.7), while taking into account that the domain of integration is $(-\infty, \infty)$, results in the following characteristic equation to determine the vibrations frequency ω ($k = \pi / \lambda$ is the wave number and λ is the half-wave length):

$$\omega^2 - \omega_0^2 - B_{0x}\frac{k(1 + kh)}{4\pi\rho}q = 0 \tag{3.1}$$

$$\omega_0^2 = \frac{Dk^4}{2\rho h}, \qquad q = B_{0x}\left[1 + \frac{c^2k(1+kh)}{4\pi i\omega\sigma h}\right]^{-1}$$

Here w_0 is the natural vibrations frequency of the plate in the absence of a magnetic field.

Let us find the magnitudes of the induced magnetic and electric field intensities in the whole space as a function of the plate deflections:

$$h_x = z\frac{qk}{h}w, \quad h_z = -ikqw, \quad e_y = -\frac{i\omega}{c}qw \tag{3.2}$$
$$h_x^{(e)} = \pm kqe^{k(h\mp z)}w, \quad h_z^{(e)} = \mp ikqe^{k(h\mp z)}w$$

Here the upper signs are taken for $z \geqslant h$ and the lower for $z \leqslant -h$.

Comparing the values of the quantities (3.2) with the corresponding values of the same quantities obtained in [1], shows that the results found in [1] agree with (3.2) for $V^2/c^2 \ll 1$ (V is the phase velocity of elastic wave propagation in the plate and c is the speed of light in vacuum). Let us also mention that (3.1) agrees to the accuracy of quantities on the order of V^2/c^2 with the corresponding characteristic equation obtained in [1] in the solution of the same problem taking account of displacement currents.

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